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Equivalence of general relativistic field theories†

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Abstract. A theorem of Aczél and Golab is strengthened and simplified. According to the new theorem any local geometric object $\psi_A(x)$ may be expressed as a differential concomitant of a suitable set of local scalars $S^B(x)$. As a consequence, it is proved that any generally covariant field theory may be described in terms of scalar fields, i.e. that the set of all general covariant field theories is equivalent to its subset of scalar field theories.

1. Introduction

In the general relativistic description of physical phenomena, as well as in the unified theoretic description of Nature, different authors apply different geometric objects as field variables, i.e. as descriptors of the physical system in question (Tonnelat 1965). For example, in his last theory Einstein applied a non-symmetric cotensor $g_{ik}(x)$, Schrödinger applied a connection $\Gamma_{kl}{}^{i}(x)$, in the consideration of the properties of the energy–momentum complex, Møller applied a system of four covectors $V_i{}^A(x)$, etc. The following questions seem to be unsolved:

(*a*) By what geometric objects Nature may be, or perhaps should be, described?

(*b*) Does there exist some object by which any physical phenomenon may be described?

In this context a theorem of Aczél and Golab is useful (Aczél and Golab 1960). This theorem states that any local geometric object $\psi_A(x)$ is equivalent to some aggregate of objects

$$\psi_A = \Psi_A(S^B, V_i{}^C, V_{ik}{}^D, \ldots). \tag{1}$$

The constituents of this aggregate are a certain number of scalars $S^B(x)$, a number of covectors $V_i{}^C(x)$, a number of objects $V_{ik}{}^D(x)$ which transform as the gradient of a covector, etc. This theorem means that anything that may be expressed by $\psi_A(x)$ may also be expressed by the aggregate $(S^B(x), V_i{}^C(x), V_{ik}{}^D(x), \ldots)$.

This theorem has not been applied a great deal, because the number of independent constituents of the above aggregate is sometimes very great. If possible, the reduction of this number would be very useful.

In the present paper we prove that the covectors $V_i{}^C$ and the objects $V_{ik}{}^D$, etc., of the above aggregate may be reduced, respectively, to the gradients of the scalars and to the second derivatives of the same scalars, etc. In this way the above theorem is strengthened and simplified. As an application of the new theorem we prove that the set of general covariant field theories is reducible to the set of scalar field theories.

By the last theorem, question (*b*) is in principle solved. By virtue of this fact the general covariant field theory may be mathematically simplified and unified.

2. The mathematical theorem

Theorem 1. Any local geometric object $\psi_A(x^k)$ may be expressed as the differential concomitant of some set of scalar fields $S^B(x^k)$.

Proof. As mentioned before, local geometric objects are the covectors V_i, connections $\Gamma_{kl}{}^{i}$, etc. In general, local geometric objects are those entities which in the case of general coordinate transformations

$$\bar{x}^k = f^k(x^l), \qquad 0 \neq \det\frac{\partial \bar{x}^k}{\partial x^l} \neq \infty, \qquad k, l = 1, 2, \ldots, m \tag{2}$$

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have definite transformation properties:

$$\bar{\psi}_A(\bar{x}) = F_A\left(\psi_B(x), \bar{x}^k, x^k, \frac{\partial x^k}{\partial \bar{x}^l}, \ldots\right), \qquad A = 1, 2, \ldots, M \tag{3}$$

where F obeys the identity and transitivity conditions (Nijenhuis 1952, Aczél and Golab 1960, Kucharzewski and Kuczma 1964, Knapecz 1968).

Let us take one arbitrary coordinate system, and let us denote the coordinates of this system by $\overset{0}{x}{}^k$. Then according to (3) any geometric object $\psi_A(x)$ in any coordinate system x^k may be expressed as follows:

$$\psi_A(x^k) = F_A\left(\overset{0}{\psi}_B(\overset{0}{x}{}^l), x^k, \overset{0}{x}{}^l, \frac{\partial \overset{0}{x}{}^l}{\partial x^k}, \ldots\right) \tag{4}$$

where $\overset{0}{\psi}_A(\overset{0}{x}{}^k)$ are the components of $\psi_A(x^k)$ in the 'zeroth' coordinate system $\overset{0}{x}{}^k$. Since, from the viewpoint of the set $\{x^k\}$ of the variable coordinate systems x^k, both $\overset{0}{\psi}_A$ and $\overset{0}{x}{}^k$ are scalars, i.e. under any coordinate transformation (2) they do not change their value:

$$\overset{0}{\psi}_A(\overset{0}{x}(\bar{x})) = \overset{0}{\psi}_A(\overset{0}{x}(x)) \tag{5}$$

$$\overset{0}{x}{}^k(\bar{x}^l) = \overset{0}{x}{}^k(x^l). \tag{6}$$

Equation (4) represents the expression of the object $\psi_A(x)$ in terms of scalar fields and their derivatives.

3. The potentials of an object

It is practical to introduce the following notation:

$$\overset{0}{\psi}_A(x) = S^A(x), \qquad \overset{0}{x}{}^l(x) = S^{A'}(x)$$

where

$$A' = M+1, M+2, \ldots, M+n. \tag{7}$$

Then expression (4) reads

$$\psi_A(x^k) = F_A\left(S^A(x^k), x^k, S^{A'}(x^k), \frac{\partial S^{A'}}{\partial x^k}, \ldots\right) \tag{8}$$

where $S^A(x)$ and $S^{A'}(x)$ are the *scalar potentials* of $\psi_A(x)$ and (8) is ψ_A expressed in terms of its potentials.

We mention without proof that the number of potentials is always less than $M+n$. For example, in four-dimensional space–time the number of scalar potentials of a covector lies between one and seven.

Some straightforward applications of this theorem may be found in the literature. For example, the Monge potentials of the velocity field (Bahr and Schöpf 1968)

$$v_k = \frac{\partial a}{\partial x^k} + b\,\frac{\partial c}{\partial x^k} \tag{9}$$

i.e. the Clebsch representation of the velocity field (Seliger and Whitham 1968), are nothing more than a special application of the above theorem.

One application of this theorem is treated in the next section.

4. The field theory theorem

Theorem 2. The set of all possible generally covariant field theories is equivalent to (or expressible by) its subset of generally covariant scalar field theories.

Proof. Any field theory is characterized by its system of equations of motion (field equations)

$$E_\alpha\left(\psi_A(x), \frac{\partial\psi_A}{\partial x^k}, \ldots\right) = 0$$

where

$$A = 1, 2, \ldots, M; \qquad \alpha = 1, 2, \ldots, M' = M, \text{ or } \neq M. \tag{10}$$

According to the principle of general covariance the expressions E_α should be covariant expressions of their arguments $\psi_A(x)$, which should be geometric objects. Since, by virtue of (8), any geometric object ψ_A is expressible in terms of its scalar potentials, the concomitant E_α is also expressible by the same set of scalar potentials. But then any field theory can be described by some aggregate of scalar fields, which are the 'ultimate' scalar potentials of the system in question.

5. Discussion

The physical significance of the last fact does not consist only in the possibility of transcribing any known vector, tensor or any other theory into scalar form, but in the possibility of considering the general and principal features of field theory: namely, it is sufficient to consider only scalar theories. The theories which are formulated in terms of scalar fields also contain everything which is contained in the non-scalar theories.

The application of other geometric objects instead of scalars in the field theory, i.e. in the general relativistic description of local phenomena, is not a necessity, but a question of *mathematical convenience* only. The theories which are formulated in terms of vector, tensor, etc., fields only reproduce (perhaps in a more compact form) what is already contained in some purely scalar theory.

By virtue of this fact (without loss of generality) in the study of the general features of the general relativistic field theories it is sufficient to consider only those theories which are formulated in terms of scalar fields.

Since any generally covariant field theory may be made scalar, the field theories may be treated in a uniform manner.

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